# *Factor Analysis in Longitudinal and Repeated Measures Studies*

Jack McArdle, Psychology Dept.,
University of Virginia, Charlottesville, VA

*The "Factor Analysis at 100" Meeting*

University of North Carolina, May 2004

## *Outline*

1. A Brief History of Longitudinal Factor Analysis
2. Modeling Trends based on Latent Growth Models
3. Two Occasion Factor Analysis
4. Two Occasion Latent Difference Scores
5. Simple Structure or Simple Dynamics?
6. Multiple Occasion Dynamic Analysis
7. Multivariate Models for Biometric Twin Data

# *1. A Brief History of Longitudinal Factor Analysis*

## *Historical Milestones in Long EFA*

- Cattell (1950) *T* and *P* technique (and Cronbach's 1963 critique)
- Baker, Tucker, Rao (1950-60s) "Latent Curves"
- Harris (1963) volume on "Measurement of Change"
- Tucker's (1966) "Three-Mode Factor Analysis"
- Horn's (1973) State Traits and Change Dimensions"
- Nesselroade (1972) Longitudinal FA & Nesselroade & Cable (1977) "Factor of Differences"
- Roseboom (1975) General Linear Dynamic FA Models
- Hakstian (1975) overview of Long FA models
- Arbuckle & Friendly (1977) "Rotation to Smooth Functions"

## *Historical Milestones in Long. CFA*

- Joreskog (1970,1974) SEM for two-occasion data and Joreskog & Sorbom's (1979) SEM for multi-occasion data
- Geweke & Singleton (1981) "Factor of Economic Time Series"
- Molenaar's (1984) "Dynamic Factor Analysis"
- Arminger's (1987) "Differential Equation Factor Model"
- Meredith & Tisak (1985, 1990) "Tuckerizing Curves"
- Browne (1990) "Structured Latent Curves"
- Oud & Jansen's (2000) "State Space Factor Model"
- Oort (2001) "SEM for Three-Mode Factor Analysis."

## *A classical bivariate Long.FA model from Joreskog & Sorbom (1975)*

145

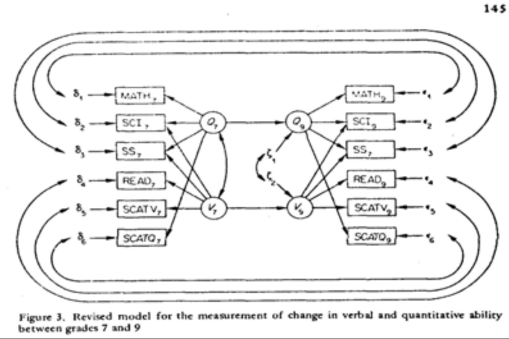



Figure 3. Revised model for the measurement of change in verbal and quantitative ability between grades 7 and 9

## *A classic univariate latent path model from Joreskog (1974)*

KARL G. JÖRESKOG 51

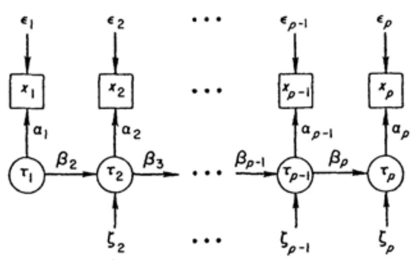



FIGURE 8.
Path analysis diagram of the quasi-Markov simplex.

## *New Dynamic SEM methods*

- Subtle differences in the choice among models define the nature of developmental process and change -- i.e., the *dynamic systems*.
- The models presented here are recent SEM extensions, but all dynamic parameters that are described in classical literature on difference score models.
- These models can be estimated using widely available techniques in SEM (LISREL, Mx, M+, AMOS).
- The traditional SEM/LISREL methods naturally provide a framework for the study of alternative dynamic models.

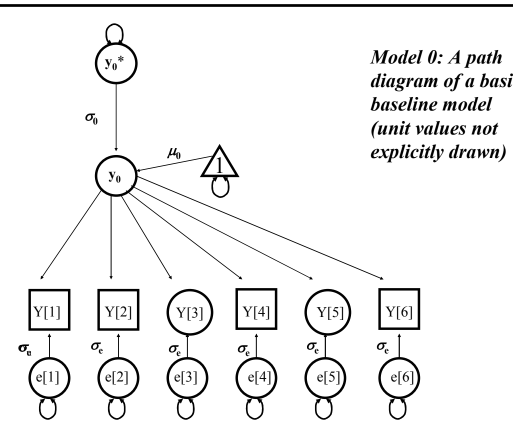
## *Evaluating Relative Goodness of Fit*

- Decisions about goodness-of-fit are relative to the data at hand -- comparisons *within* a data set can be informative, and absolute rules for goodness-of-fit indices are useful but wrong – i.e., not even the popular "non-significant $\chi^2$," or "smallest AIC" or "*RMSEA* < .05"!
- It follows that substantive knowledge is always needed.
- In fitting any factor, it is useful to recall: "Factors in a factor analysis are not *things*, but they are our evidence for the *existence of things*." (R.B. Cattell, 1950).
- Kaiser (1976) on Lawley & Maxwell (1971) – "Elegant, delightful …but don't take it seriously!"

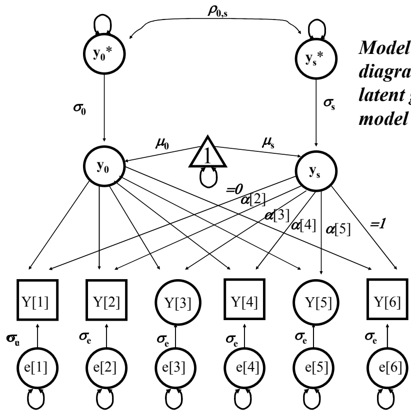
# *2. Modeling Trends Based on Latent Growth Models*

## *Fitting SEM-HLM Mixed Effects with an Age Basis + Practice*

- A set of nested models are defined to evaluate changes using all data nested within the same person (and spouses)
- Model 0 is "no change over time"
  $\boldsymbol{Y[t]_n = y_{0n} + e[t]_n}$
- Model 1 is change is "linear or exponential with age"
  $\boldsymbol{Y[t]_n = y_{0n} + A[t]\ y_{sn} + e[t]_n}$ where $A[t] = \{ \Sigma_{1,t}\ \delta \}$
- Model 2 is a "two-part linear spline with a knot age $\tau$,"
  $\boldsymbol{Y[t]_n = y_{0n} + A1[t]\ y_{1n} + A2[t]\ y_{2n} + e[t]_n}$
  where $A1[t]$ = T-$k$ *iff* t< $\tau$, and $A2[t]$ = t- $\tau$ *iff* t> $\tau$.
- Model 3 is change is "with age and practice/exposure"
  $\boldsymbol{Y[t]_n = y_{0n} + A[t]\ y_{sn} + e[t]_n + \Pi[r]}$. {0 or 1} ,
- Statistical estimates via maximum likelihood and goodness-of-fit via likelihood ratio tests for pairs

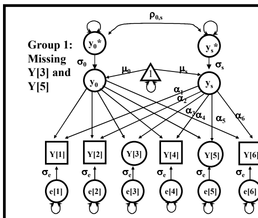


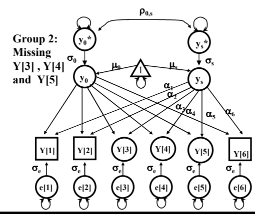

***Model 0: A path diagram of a basic baseline model (unit values not explicitly drawn)***

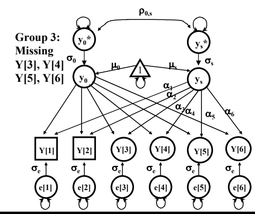



***Model 1: A path diagram of a basic latent growth model with slopes***

***Multiple groups with different patterns of incomplete data.***
***(from Horn & McArdle, 1980)***



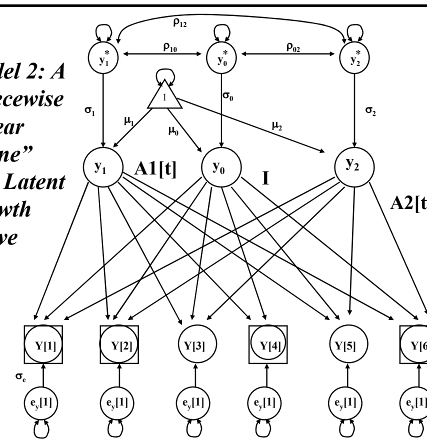




## *The "piecewise" latent growth model*

- A curve curve can be expressed by defining a cutting score or "knot-point" $\tau$, where
  $Y[t]_n = y_{0\,n} + \boldsymbol{A1}[t]\, y_{1n} + \boldsymbol{A2}[t]\, y_{2n} + e[t]_n$
  where $\boldsymbol{A1}[t] = T - k$ *iff* $t < \tau$,
  and $\boldsymbol{A2}[t] = T - k$ *iff* $t > \tau$.

- *This model implies that:*
  $y_0$ is the *intercept* term,
  $y_1$ is the *pre-knot slope* term *before* $\tau$,
  $y_2$ is the *post-knot slope* term *after* $\tau$, and
  $e[t]$ is the error term around the curve.

***Model 2: A "Piecewise Linear Spline" as a Latent Growth Curve***

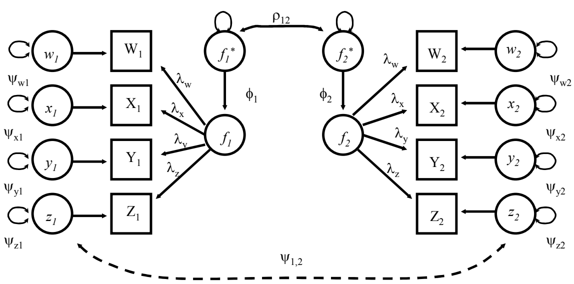



# *3. Two Occasion Longitudinal Factor Analysis*

## *Longitudinal Factor Model Questions*

- Can we consider the same numbers of common factor model over time, for occasions *1 and 2*, does
  $\boldsymbol{Y^*}[t]_n = \boldsymbol{\Lambda}[t]\ \boldsymbol{f}[t]_n + \boldsymbol{u}[t]_n$ ?
- Some questions are about the loadings $\boldsymbol{\Lambda}[t]$ over time -- For all *T,* does $\boldsymbol{\Lambda}[1] = \boldsymbol{\Lambda}[2]$?
- Other questions are about factor score $\boldsymbol{f}[t]_n$ over time -- For all *N*, does $\boldsymbol{f}[1]_n = \boldsymbol{f}[2]_n$*?*
- Each set of model restrictions deals with a different question about "construct equivalence"

## *A standard factor invariance over time model applied to scale-level data*




## *Invariance of the Factor Loadings*

- The loadings factor loadings $\Lambda$[t] are latent variable regression coefficients given the common factor score is the independent variable and the observed scores are the set of dependent variables.
- These $\Lambda$[t] are used to define the meaning and the name we assign to the common factor based on the pattern of relationships with the observed variables.
- It follows that questions about whether or not we have measured "the same factor over time" are based solely on $\Lambda[1] = \Lambda[2]$?

## *Invariance of the Factor Scores*

- The common factor scores ***f***[t] are not directly observed, but their variances ($\phi[t]^2$) and covariances ($\phi$[t,t+1]) define the relative position of each individual on the factor scores.
- We can question the relative size of the variances over time as $\phi[t]^2 = \phi[t+1]^2$ ?And we can question the size of the correlation of the factor scores over time

  $\rho[t,t+1] = \phi[t,t+1] / (\phi[t]\ \phi[t+1])^{1/2} = 0$?
- But whether or not we have "the same factor over time" is *not in any way based on the factor score parameters*!

## *Factorial Invariance under Selection*

- In a seminal series of papers, W. Meredith's (1964-65) extended Lawley's (1941) selection theorems to the common factor case and demonstrated:

  IF

  (1) a factor model $\Sigma = \Lambda\Phi\Lambda' + \Psi^2$ holds in a population,

  (2) samples are selected from that population in any way, randomly or non-randomly,

  THEN

  (3) the factor loadings $\Lambda$ will remain invariant,

  (4) but the factor variances and covariances $\Phi$ will not remain invariant.
- Under these premises, a reasonable search for factor loading invariance over time ($\Lambda$[t]) should be done without any restrictions on the factor covariances over time ($\phi$[t,t+1]).
- Under any form of sample selection, iIt is generally unreasonable to fix $\phi[t,t+1]=0$.

## *Configural vs Metric Invariance (Horn & McArdle, 1980, 1992)*

- Previous research based on Thurstone (1947) has defined different levels of factor loading invariance:
  - *Configural Invariance* is defined as exact equality of all positions of the elements $p\{\Lambda[1]\} = p\{\Lambda[2]\}$ (i.e., the same path diagram).
  - *Metric Invariance* is defined as the exact equality of every element in $\Lambda[1] = \Lambda[2]$.
  - *Partial Invariance* is defined as the exact equality of some elements in $\Lambda[1] = \Lambda[2]$.
- Fundamental question of equality of measurement over time are embedded in these comparisons.

## *Invariance without Standardization*

- Standardization of observed scores is not desired because invariance is a raw score regression problem.
- If standardization is desired, it must be done *using the same mean and standard deviation* for all occasions;

  $z[1]_n = ( y[1]_n - \mu ) / \sigma$

  $z[2]_n = ( y[2]_n - \mu ) / \sigma$
- This mean and standard deviation could come from, say, the time one scores:

  $z[1]_n = ( y[1]_n - \mu[1]) / \sigma[1]$

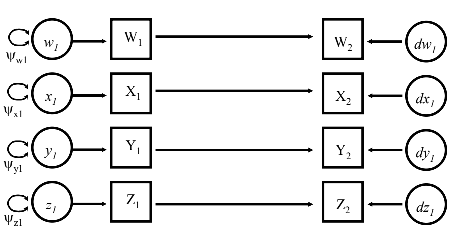
  $z[2]_n = ( y[2]_n - \mu[1]) / \sigma[1]$
- But, as in raw-score regression, questions about the "equality of coefficients" are not possible to answer if the observed variables have been standardized within each occasion because information has been lost.

# *4. Analyses based on Longitudinal Difference Scores*

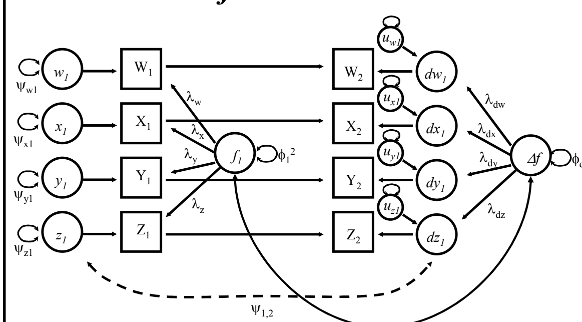
### *From McArdle & Nesselroade (1994)*

- A simple set of difference scores are written
  $d(w)_n = \boldsymbol{W}[2]_n - \boldsymbol{W}[1]_n$ or $\boldsymbol{W}[2]_n = \boldsymbol{W}[1]_n + d(y)_n$
  $d(x)_n = X[2]_n - X[1]_n$ or $\boldsymbol{X}[2]_n = \boldsymbol{X}[1]_n + d(x)_n$
  $d(y)_n = \boldsymbol{Y}[2]_n - \boldsymbol{Y}[1]_n$ or $\boldsymbol{Y}[2]_n = \boldsymbol{Y}[1]_n + d(y)_n$
- The use of *fixed unit weights* now permits a variety of reliability and stability models because we can fit
  $d(m)_n = \Lambda \mathbf{d}(m)\, \Delta f_n + d(um)_n$
- This d-form does not alter the interpretation or statistical testing of factor invariance over time:
  if $\Lambda[1] = \Lambda[2]$ then $\Lambda[1] = \Lambda d$ or $\Lambda b = \Lambda w$
  or, the factor pattern *between* variables equals the factor pattern *within* variables.
- This result is consistent with previous work on this topic (see Nesselroade & Cable, 1977).

### *Adding difference scores to a model for score-level data*



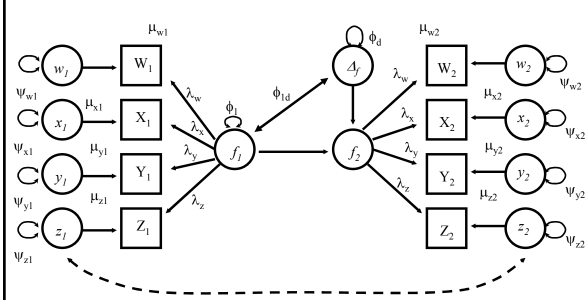

### *A "factors of difference scores" model for score-level data*



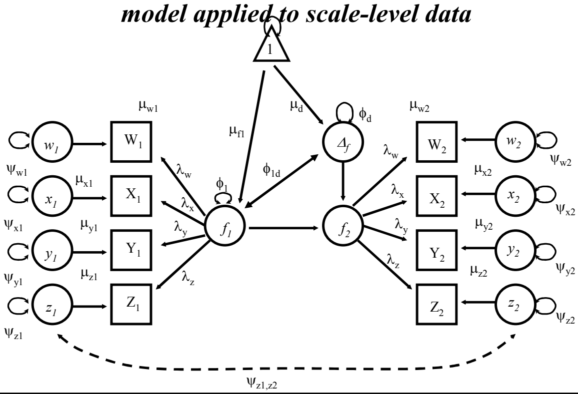

### *Other SEMs can also be explored?*

- If factorial invariance can be achieved, a simple latent difference score can be written at the factor score level
  $\Delta f_n = f[2]_n - f[1]_n$ or $f[2]_n = f[1]_n + \Delta f_n$
- This model requires special constraints--- the loadings and mean intercepts are forced to be equal over time, and all differences are accounted for by the factors.
- The same factor is used to account for both the *covariance* differences and the *mean differences* over time, so this can alter the interpretation or testing of factor invariance over time.

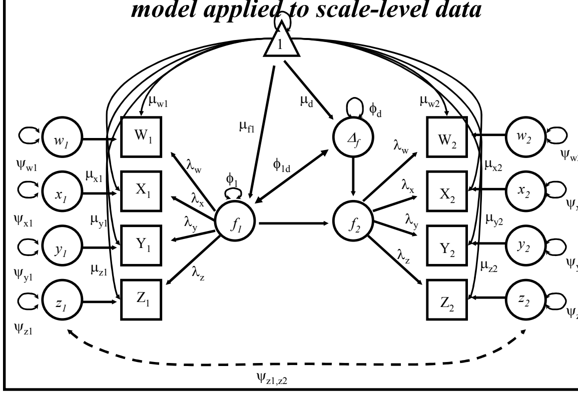
### *A "differences in common factor scores" model applied to scale-level data*



### *A “differences in common factor scores” model applied to scale-level data*




### *A “differences in common factor scores” model applied to scale-level data*




# *5. Simple Structure OR Simple Dynamics?*

## *What if Factorial Invariance does not fit?*

- The basic ideas of *metric factorial invariance* are a primary consideration in longitudinal SEM.
- In the absence of metric factorial invariance it is hard to:
  (a) assert the same factors are measured with the same variables (i.e., “apples and oranges”)
  (b) go any further with growth and change models (i.e., “rubber rulers”).
- Thus, it seems reasonable for a researcher to examine many ways to achieve factorial invariance, even at the cost of factorial complexity (i.e., “life is not simple”)

## *Thurstone (1935, 1947) and “Simple Structure” Rotation*

- "One of the turning-points in the solution of the multiple factor problem is the concept of "simple structure". It will be shown that this concept enables us to obtain an invariance of factorial description that has not, so far, been available by other means.…”
- “When a factor matrix reveals one or more zeros in each row, we can infer that each of the tests does not involve all the common factors that are required to account for the intercorrelations of the battery as a whole. This is the principle characteristic of a simple structure. (Thurstone, 1947, p.181).

## *Thurstone (1935, 1947) and “Simple Structure” Rotation*

- The factorial description of a test must remain invariant when the test is moved from one battery to another which involves the same common factors. ”
- “The factorial composition of a set of primary factors that have been found in a complete and overdetermined simple structure remains invariant when the test is moved to another battery involving the same common factors and in which there are enough tests to make the simple structure complete and overdetermined.”
- Thurstone (1947, p.365) also clearly distinguished "configurational invariance” from “metric invariance.”

## *Cattell's (1944) Alternative -- Simultaneous Simple Structure*

- "The principle of parsimony, it seems, should not demand "Which is the simplest set of factors for reproducing this particular correlation matrix?" but rather "Which set of factors will be most parsimonious at once with respect to this and other matrices considered together?"
- ... The criterion is then no longer that the rotation shall offer the fewest factor loadings for any one matrix; But that it shall offer fewest dissimilar (and therefore fewest total) loadings in all of the matrices together. ...

## *Cattell's (1944) Alternative -- Simultaneous Simple Structure*

- To indicate the historical foundations from which it builds, however, and the fact that it extends to several matrices simultaneously ... it might equally well be called "simultaneous simple structure." (Cattell, pp.273-274).
- "Parallel Proportional Profiles" (1944)
- The "Confactor solution" (1955)
- Meredith (1964) showed covariances were required for "parallel proportional profiles"
- McArdle & Cattell (1988, 1994) provided conditions for SEM identification of "confactor rotation" with multiple groups.

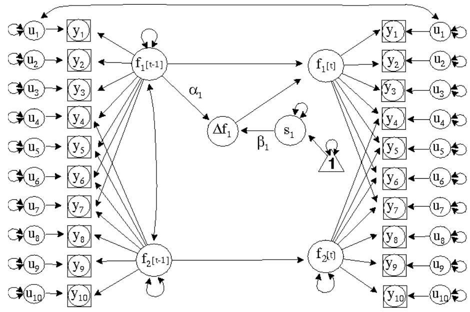
## *McArdle & Cattell (1994) MBR*

- The minimum requirement to uniquely identify a *k*-factor factor model in one group is to place $k^2$ fixed constraints in $\Lambda$ and $\Phi$.
- Typically *k* constraints are placed on $diag(\Phi)^{(1)} = \mathbf{I}$ and $k(k-1)$ are placed on the columns of $\Lambda$. *Same as in multiple groups*.
- Alternatively, it is also possible to identify the MG model by only requiring $\Phi^{(1)} = I$, $\Phi^{(2)} = diag(\phi^2)$, with $\Phi^{(g>2)} = free$, and $\Lambda = free$

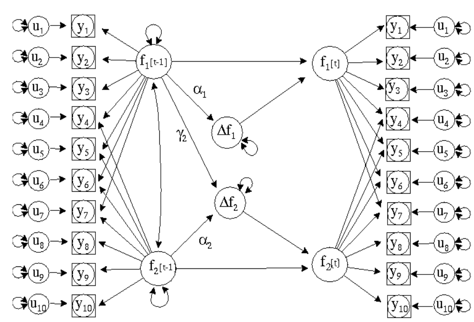
## *McArdle & Nesselroade (1998) longitudinal extensions of Confactor*

- Typically *k* constraints are placed on $diag(\Phi[1,1]) = \mathbf{I}$ and $k(k-1)$ are placed on the columns of $\Lambda$. *Same as in multiple groups*.
- As before, the model is identified by only requiring $\Phi[1,1] = I$, $\Phi[2,2] = diag(\phi^2)$, $\Phi[t,t] = free$ (t>2), and $\Lambda = free$.
- Thus, in repeated measures data the additional constraints for identification can be among the *across-time* $\Sigma[t,t+j] = \Lambda\, \Phi[t,t+j]\, \Lambda'$

## *Two Factors ID by Simple Growth*




## *Two Invariant Factors ID by Simple Lead-Lag*



## *Future issues in "Dynamic But Structural Modeling"*

- Current methodological and substantive research is needed for dealing effectively with *more than one* dynamic variable.
- Using models from *linear dynamics* combined with SEM using "difference" $\Delta Y/\Delta t$ or "differential" $dY/dt$.
- Further possibility of combining a change model $\Delta Y/\Delta t$ together with measurement.
- Further consideration of "Rotation to Simple Dynamics" (after McArdle, 1985; McArdle & Nesselroade, 1998)

# *6. Models for Multiple Occasion Longitudinal Data*

## *Invariance over Multiple Occasions*

- We first consider the same numbers of common factor model over time, for occasions *T=6, does*

$$\boldsymbol{Y^*}[t]_n = \Lambda\ \boldsymbol{f}[t]_n + \boldsymbol{u}[t]_n \text{ ?}$$

- One reasonable way evaluate testing of factor invariance over time is to simply examine

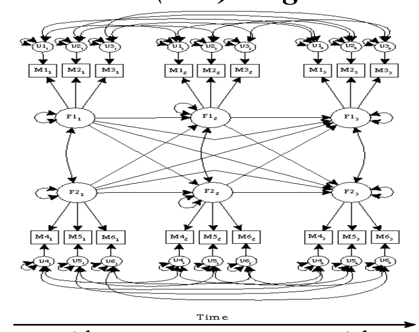
$$\Lambda b[t] = \Lambda w[t] \text{ or,}$$

the factor pattern *between time* equals the factor pattern *within time* (see Nesselroade & Cable, 1977).

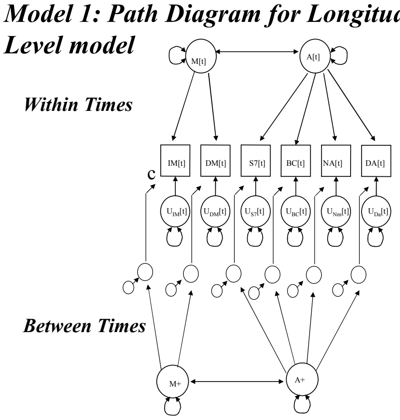
- If metric factorial invariance can be achieved, the latent difference score can be written at the factor score level

$$\Delta \boldsymbol{f}[t]_n = \boldsymbol{f}[t+1]_n - \boldsymbol{f}[t]_n \quad \text{or} \quad \boldsymbol{f}[t+1]_n = \boldsymbol{f}[t]_n + \Delta \boldsymbol{f}[t]_n$$

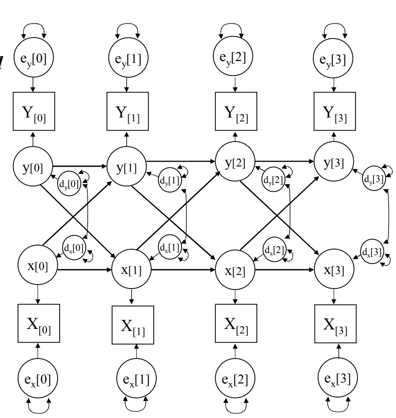
## *More possibilities but similar issues with extended (T>2) longitudinal data*



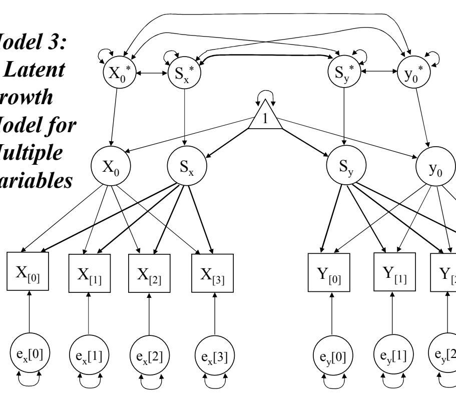

## *Model 1: Path Diagram for Longitudinal Multi-Level model*




*Model 2: Cross-lagged Latent Regression Model With Invariant Factor Patterns*

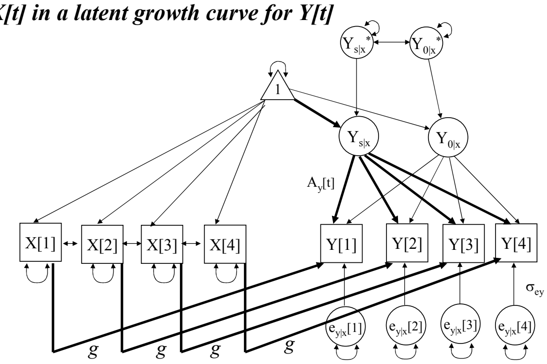


***Model 3: A Latent Growth Model for Multiple Variables***

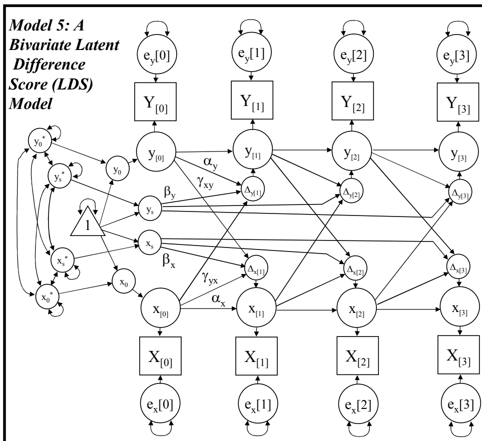



***From McArdle (1988, 1989)***

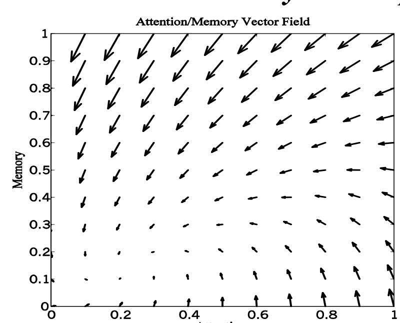
***Model 4: A time varying covariate X[t] in a latent growth curve for Y[t]***




***Model 5: A Bivariate Latent Difference Score (LDS) Model***




## *A vector field representation of the Model 5 latent variable dynamic equations*




# *7. Multivariate Models for Biometric Twin Data*

## *Repeated Measures Models for Twins*

- The biometric analysis of twin data require important features beyond other forms of repeated measures SEM.
- McArdle, Goldsmith & Connell (1980; *Behavioral Genetics*) showed how SEM programs like LISREL could be use to carry out multivariate twin analyses.
- McArdle (1986; *BG*) showed how latent growth models could be used with longitudinal twin data.
- McArdle & Goldsmith (1990; *BG*) showed why the inclusion of a common factor model was useful in all multivariate twin analyses.
- McArdle & Hamagami (2003; *BG*) described SEM models for evaluating dynamic concepts within longitudinal twin analyses.

***Figure 1: Basic Univariate Twin Models as SEM -- Note inclusion of constant (McArdle, 1986)***

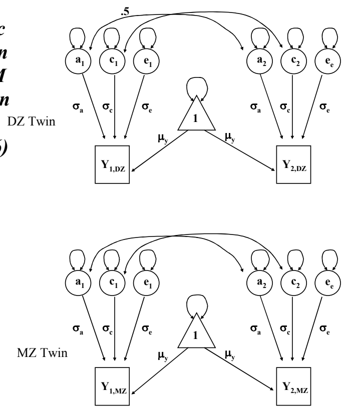



***Fig. 2: Two key common factor alternatives for twin models***

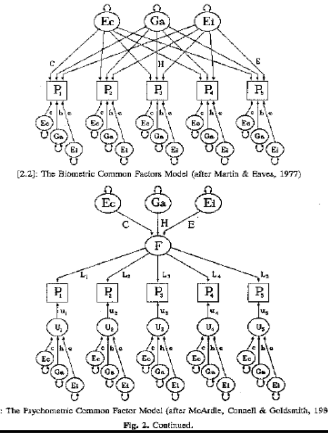



***Figure 2: Adding a latent growth model to the twin design***

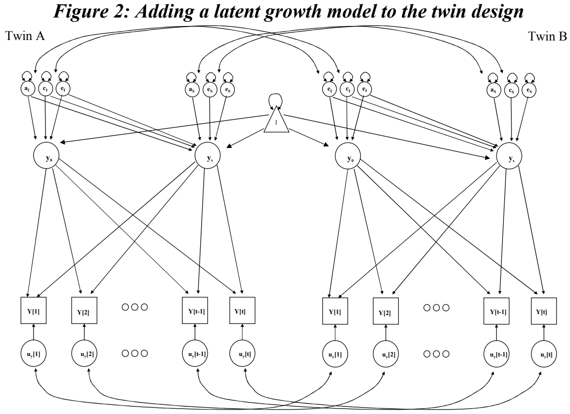
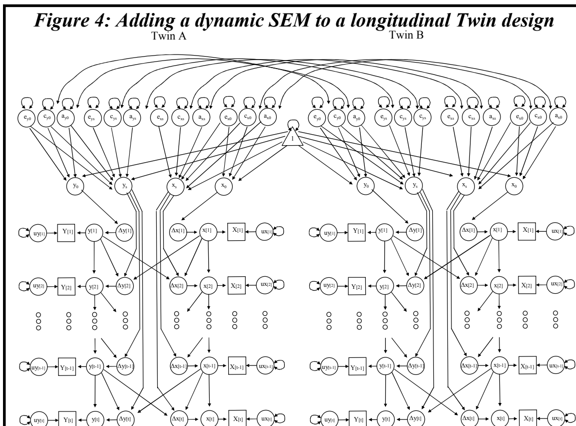



***Figure 4: Adding a dynamic SEM to a longitudinal Twin design***




# *Recent References*

**Available from the Jefferson Psychometric Laboratory**
**(http:\\ kiptron.psyc.virginia.edu)**

McArdle (2000). A dynamic but structural equation model. *Feschrift for Karl Joreskog*. SSI, Chicago.

Hamagami & McArdle (2001). In Marcoulides, *Advanced structural equation models*.

McArdle, Hamagami, Meredith & Bradway (2001). *Learning and Individual Differences*.

McArdle & Nesselroade (2003). *Growth Curve Analysis in Psychological Research*. In Velicer et al.

McArdle & Hamagami (2003). Structural equation models for evaluating dynamic concepts within longitudinal twin analyses. *Behavioral Genetics*.

McArdle, Hamagami, Jones ….& Albert (2004). *Jour. of Gerontology: Psychological Sciences*.

# *Acknowledgements*

- NIA Grants -- AG02695, AG4704, Current Grant AG07137

- John L. Horn, USC
- John Nesselroade, UVa
- William Meredith, UCB
- Raymond Cattell, UI-UH
- Rod McDonald, UI
- Michael Browne, OSU
- Richard Woodcock, MLC
- Fumiaki Hamagami, UVa