Developments in the Factor Analysis of Individual Time Series

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Factor Analysis at 100: Historical Developments and Future Directions
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Outline

- Cattell's Original Contribution
- T.W. Anderson's Comments
- Two Dynamic Factor Analysis Models
- Two Approaches to Estimation
- Recent Developments

Origins of the P-Technique

- First carried out, named and promoted by Raymond Cattell
  - Factor analysis is carried out on one person at a time.
    - Repeated observations on tests over time on the same person are analyzed in the same way as repeated observations on tests over different persons at the same time.
    - Correlated and factor analyzed in the usual manner.
    - No special-purpose computational methodology required.
- Many applications since then. Reviews:
  - Luborsky, L., & Mintz, J. (1972)
  - Jones & Nesselroade (1990)

Initial Aims of P-Technique

- Cattell, Cattell & Rhymer (Psychometrika, 1947) "P-Technique Demonstrated . . ." (Self Ratings)
  - First empirical application
- Questions to be answered using P-technique:
  - Does a factor analysis on an individual yield a similar factor pattern to that obtained from a large group of persons?
  - To what extent do factor patterns obtained from different individuals differ.
- No mention was made at this stage of lagged correlation or time series.

Early Reactions to P-technique

- T. W. Anderson (*Psychometrika*, 1963).
  - Pointed out that information about change over time is neglected
  - Two stage procedure: Factor analysis + Time series analysis of factor score estimates.
  - Difficulty: Factor scores and factor variables do not have the same time series.
  - Implied that factors represented a latent time series causing concurrent and lagged correlations amongst manifest variables
  - Impetus to later work on factor analysis of individual time series
    - Maximum likelihood estimation. No factor scores.
- Wayne Holtzman (1963, Harris volume)
  - Pointed out that lagged correlations and cross-correlations are disregarded.

Influence on Cattell

- Cattell (1963, Harris volume on measurement of change)
  - Change over time is now given specific attention. Comments by Anderson and Holtzman are discussed.
  - Rather than considering the effects of lagged shocks on current variables which is usually done in time series, Cattell perceived them as lead effects of current shocks on future variables.
    - Mathematically equivalent when the time series is stationary, but not conceptually equivalent.
    - Stated that factors could be conceived of as being dependent or independent: "... among factors some are dependent and some are independent" (Cattell,1963, p. 188)
    - Implied that factors could be latent shock variables influencing future manifest variables at various leads: "The aim is now to discover the interval between a change in the factor level and the change in the variable level which it causes." (Cattell,1963, p. 188)

Emergence of Dynamic Factor Analysis

- The Cattell (1963) chapter and Anderson (1963) article served as impetus to the development of methodology for the factor analysis of individual time series.
- Dynamic factor analysis: Factor analysis model with a few factors that account for change in a larger number of manifest variables over time.
- Two main types of dynamic factor analysis (cf. Browne & Nesselroade, McDonald Festschrift)
  - Process Factor Analysis Model: Factors follow a stationary vector time series that induces a time series for manifest variables.
  - Shock Factor Analysis Model: Factors are random shocks that are distributed independently across time but may be concurrently correlated.

Process Factor Analysis (PFA) Model

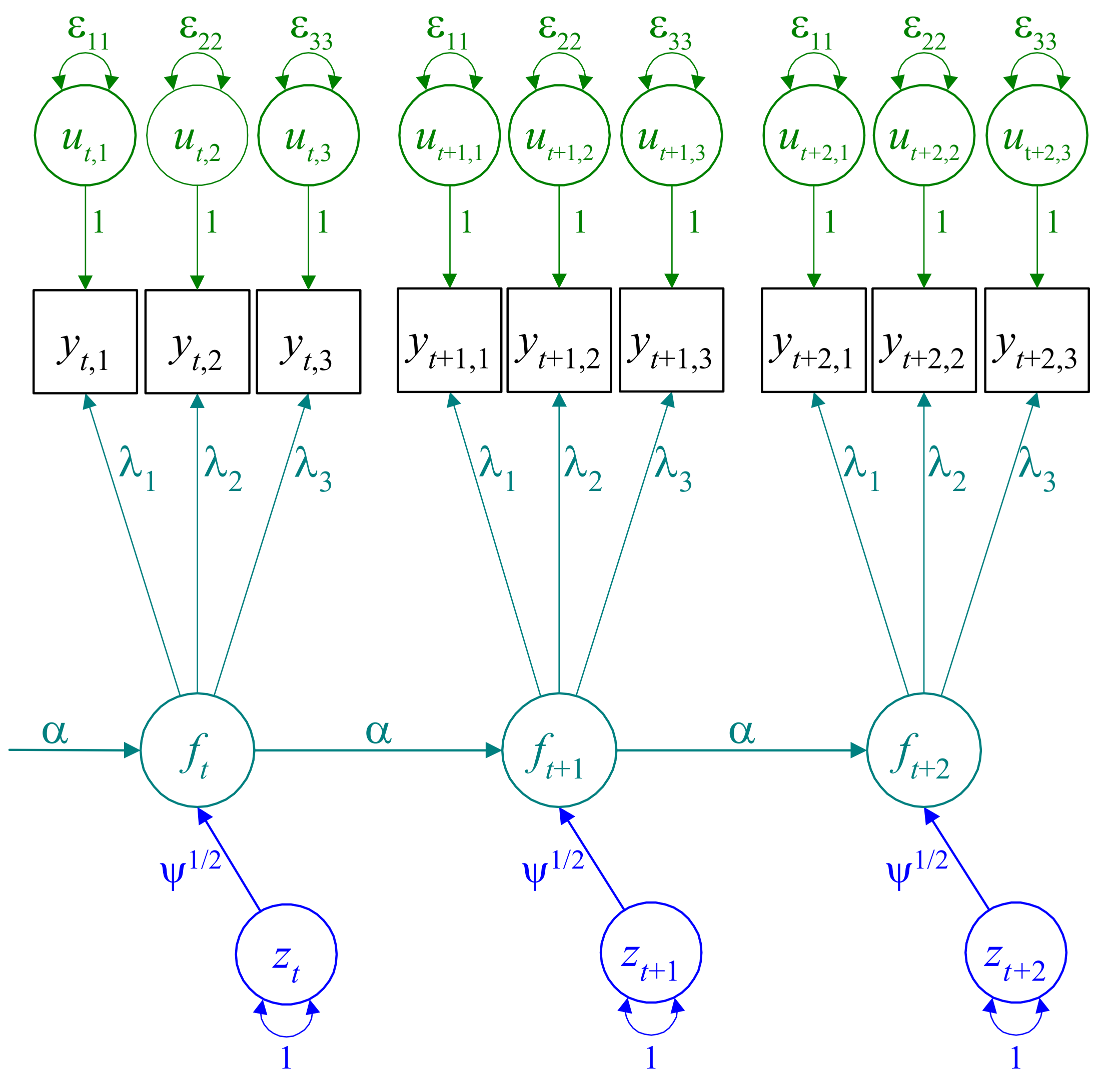
- Influenced by Anderson's view of factors following a VARMA time series.
- Key contributions: Engle & Watson (1981), Immink (1986), Nesselroade, McArdle, Aggen & Meyers (2002)
- Data model:
  - Factor analysis model for manifest variables:

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{\Lambda} \boldsymbol{f}_t + \boldsymbol{u}_t$$

  - VARMA(p, q) time series model for factors:

$$\boldsymbol{f}_t = \sum_{\ell=1}^{p} \boldsymbol{A}_\ell \boldsymbol{f}_{t-\ell} + \boldsymbol{z}_t + \sum_{\ell=1}^{q} \boldsymbol{B}_\ell \boldsymbol{z}_{t-\ell}$$

PFA(1, 0) Model



ε11 ε22 ε33 ε11 ε22 ε33 ε11 ε22 ε33
u t,1 u t,2 u t,3 u t+1,1 u t+1,2 u t+1,3 u t+2,1 u t+2,2 u t+2,3
1 1 1 1 1 1 1 1 1
y t,1 y t,2 y t,3 y t+1,1 y t+1,2 y t+1,3 y t+2,1 y t+2,2 y t+2,3
λ1 λ2 λ3 λ1 λ2 λ3 λ1 λ2 λ3
α α α
f t f t+1 f t+2
ψ1/2 ψ1/2 ψ1/2
z t z t+1 z t+2
1 1 1

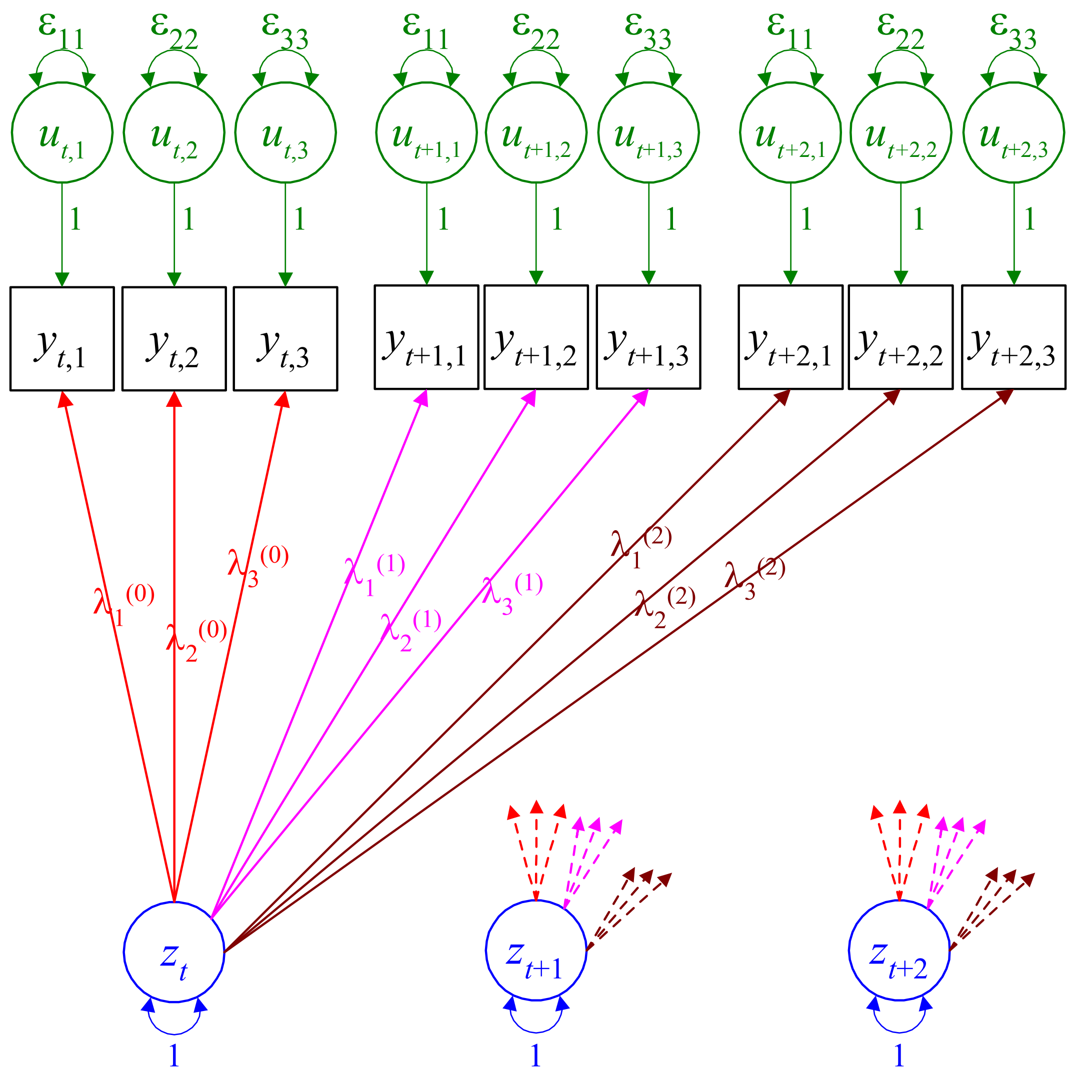
Shock Factor Analysis (SFA) Model

- Concordant with Cattell's view that shock factors influence future manifest variables at various time points.
- Key contributions: Geweke & Singleton (1981), Molenaar (1985)
- Data model $SFA(q^*)$

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \sum_{\ell=0}^{q} \boldsymbol{\Lambda}_\ell \boldsymbol{z}^*_{t-\ell} + \boldsymbol{u}_t$$

Thus $\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_{q^*},$ represent the effect of a shock factor vector variable, $\boldsymbol{z}_t,$ on the manifest vector variable, $\boldsymbol{y}_t,$ at the same time period, one time period in advance, two time periods in advance, $\ldots\, q$ time periods in advance.

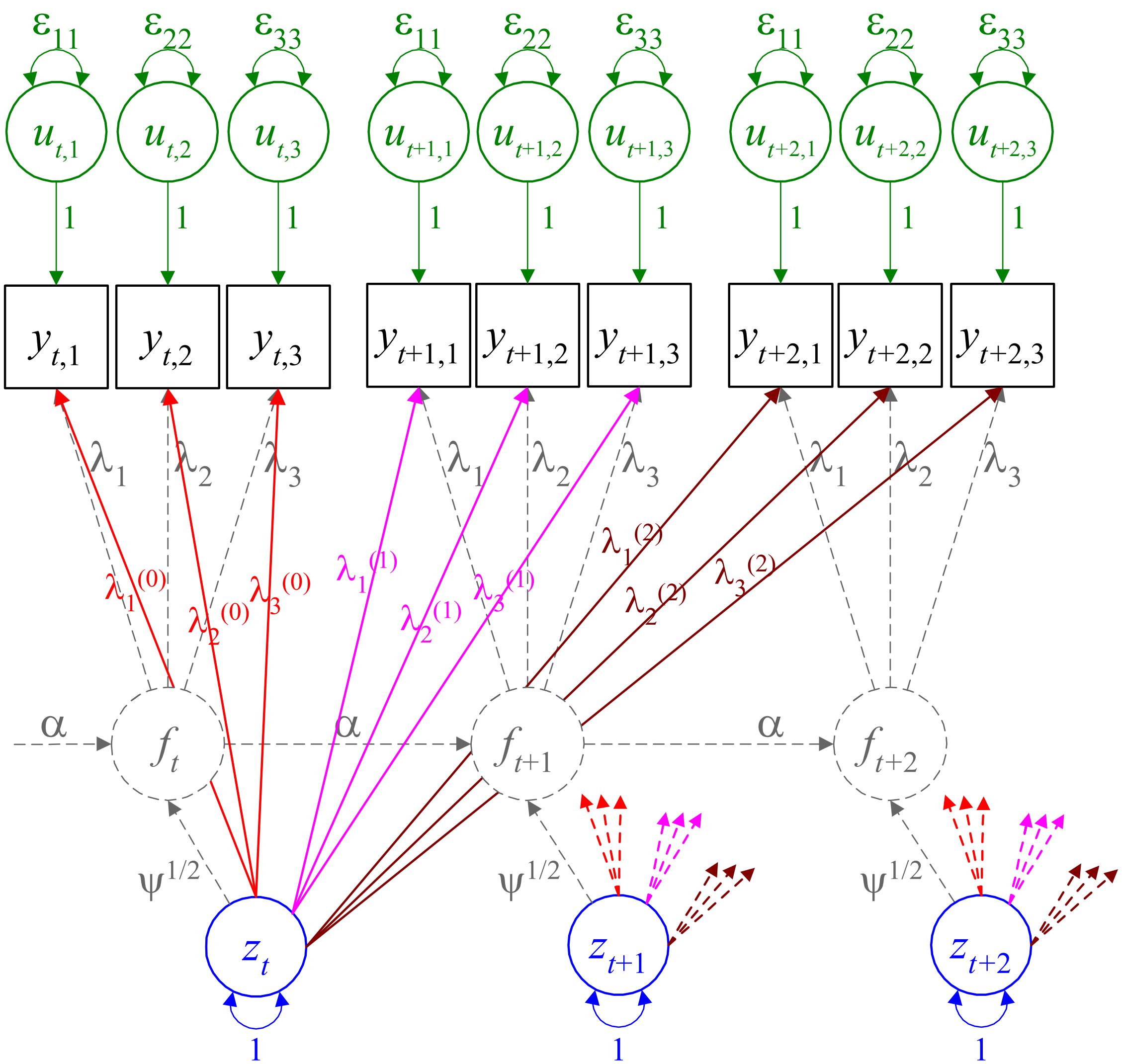
Shock Factor Analysis Model



Relationships Between PFA and SFA

- In PFA the emphasis is on a latent time series representing a dynamic psychological process.
- In SFA the emphasis is on latent shocks driving a manifest time series.
- Any PFA(p, q) model may be approximated arbitrarily closely by a SFA(q^*) model if q^* is chosen to be large enough. The approximating model will have substantially more parameters.
- It is straightforward to determine the SFA(q^*) model that approximates a given PFA(p, q) model. The SFA(q^*) model represents the implied effect of the random shocks of the PFA(p, q) model on the manifest variables.

Relationship Between PFA and SFA



Two Approaches to Estimation - I

There are two main approaches to estimation in dynamic factor analysis.

- Full Information Maximum Likelihood. Engle & Watson (1981), Immink (1986).
  - A likelihood function based on the raw data is maximized. (Maximum Normal Likelihood)
    - Estimators with known asymptotic properties are obtained.
    - Computationally intensive. A large data set is stored and manipulated by the program.
    - No program is currently available.

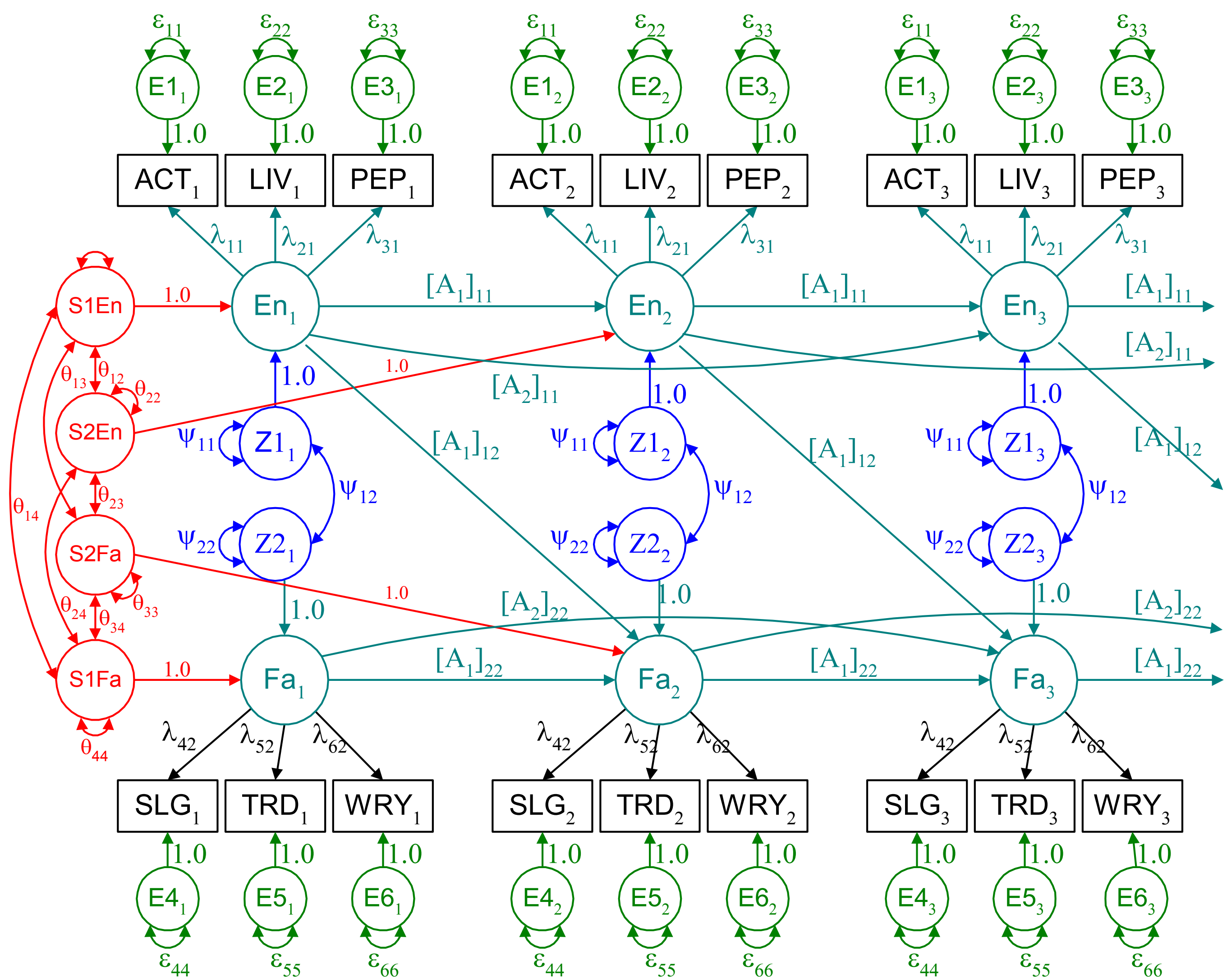
Two Approaches to Estimation - II

- Two stage procedure. Calculate lagged correlation matrix and apply standard structural equation modeling software to obtain parameter estimates. Molenaar (1985), Nesselroade, McArdle, Aggen & Meyers (2002).
  - Can be used in practice now!
  - Provides convenient residuals.
  - Maximum likelihood estimation with its associated statistical theory is not available.
    - Maximum Wishart likelihood is often used but the likelihood function is inappropriate and no associated statistical properties are known.

Stationarity Assumption

- Fundamental assumption for use of lagged correlation coefficients
  - Stationarity
    $$\text{Cov}(y_1, y_2) = \text{Cov}(y_2, y_3) = \text{Cov}(y_3, y_4) = \cdots$$
    $$\text{Cov}(y_1, y_{1+\ell}) = \text{Cov}(y_2, y_{2+\ell}) = \text{Cov}(y_3, y_{3+\ell}) = \cdots \; \forall \ell \geq 0$$
  - Implies $\rho_{12} = \rho_{23} = \rho_{34} = \cdots$
    $$\rho_{13} = \rho_{24} = \rho_{35} = \cdots$$
    $$\rho_{14} = \rho_{25} = \rho_{36} = \cdots$$ etc.
- For consistency of assumptions any time series model fitted to lagged correlations should be constrained to be stationary.

Path Diagram of a PFA(2,0) Model



Current Research Aims

- To develop special purpose methodology for process factor analysis of lagged correlation matrices.
- To use this methodology to provide special purpose software for fitting PFA models to lagged correlation matrices.
  - This will improve on software developed primarily for other purposes by :
    - Ensuring stationarity of the fitted model
    - Providing additional facilities - lagged factor correlations, show whether or not autoregressive and moving average weights satisfy invertibility conditions, breakdown of fit at different lags, show indirect effects of shocks on manifest variables, exploratory rotation of factors

The Correlation Structure

- The correlation structure employed is

$$\boldsymbol{P}_0 = \boldsymbol{\Lambda}\boldsymbol{\Phi}_0\boldsymbol{\Lambda}' + \boldsymbol{D}_\epsilon$$
$$\boldsymbol{P}_\ell = \boldsymbol{\Lambda}\boldsymbol{\Phi}_\ell\boldsymbol{\Lambda}'$$

  where the $m \times m$ lagged factor correlation matrix $\boldsymbol{\Phi}_\ell = \text{Cov}(\boldsymbol{f}_{t+\ell}, \boldsymbol{f}_t')$ is a function of the $m \times m$ autoregressive weight matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$, the moving average weight matrices $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots$, the random shock covariance matrix, $\boldsymbol{\Psi}$ and an initial state covariance matrix $\boldsymbol{\Theta}$. Details are omitted to avoid presenting a mass of algebra.

  - $\boldsymbol{\Phi}_0$ is a symmetric matrix with unit diagonals
  - $\boldsymbol{\Phi}_\ell,\ l = 1, 2, \ldots$ are nonsymmetric square matrices

Summary of Parameter Matrices

- Free parameters
  - $\boldsymbol{\Lambda}$ — $k \times m$ Factor matrix
  - $\boldsymbol{D}_\epsilon$ — Diagonal $k \times k$ error covariance matrix
  - $\boldsymbol{A}_i$ — $m \times m$ Autoregressive weight matrix
  - $\boldsymbol{B}_j$ — $m \times m$ Moving average weight matrix
  - $\boldsymbol{\Psi}$ — $m \times m$ symmetric random shock covariance matrix
- Number of identification conditions $= m^2$ as in standard factor analysis.
- Nonlinear matrix functions of free parameters
  - $\boldsymbol{\Phi}_\ell$ — $\ell = 0, 1, \ldots, L : m \times m$ lagged factor correlation matrices. Nonlinear constraints are applied to $\boldsymbol{\Phi}_\ell$ to yield unit diagonals.
  - $\boldsymbol{\Theta}$ — $sm \times sm$ covariance matrix of the s initial state vectors. $(s = \max(p, q))$

Two Sets of Regression Weights in PFA

- In Classical Factor Analysis there is a single system of regression equations with the factor loadings in $\mathbf{\Lambda}$ as regression weights.
- In Process Factor Analysis there are two systems of regression equations
  - Factor Analysis model with factor loadings in $\mathbf{\Lambda}$ as regression weights
  - Autoregression with elements of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ as regression weights plus moving average weights $\boldsymbol{B}_1, \boldsymbol{B}_2 \ldots$
- Two latent variable covariance matrices: $\mathbf{\Psi}$, $\mathbf{\Phi}$
- In exploratory PFA the matrices $\mathbf{\Lambda}, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots,$ $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots,$ $\mathbf{\Psi}, \mathbf{\Phi}$ are all affected by rotation of factors.

Ordinary Least Squares Estimation

- Lagged correlation matrices $(k \times k)$: $\boldsymbol{R}_0, \boldsymbol{R}_1, \ldots, \boldsymbol{R}_L$
- Implied covariance matrices $(k \times k)$: $\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_L$
- Discrepancy function to be minimized. Overall residual sum of squares,

$$F = \sum_{\ell=0}^{L} F_\ell$$

where

$$F_\ell = \sum_{i=1}^{k} \sum_{j=1}^{k} [\boldsymbol{R}_\ell - \boldsymbol{P}_\ell]_{ij}^2$$

is the residual sum of squares at lag ℓ and $L \geq p = 2$ is the maximum lag considered. (e.g. $L = 2$ or 3 or 4)

Starting Values

- Convergence of the Gauss-Newton algorithm can be problematical if poor starting values are used.
- Algorithm used for obtaining starting values:
  - OLS factor analysis of $\boldsymbol{R}_0$ to yield $\widetilde{\boldsymbol{\Lambda}}$ and $\widehat{\boldsymbol{D}}_\epsilon$
  - Oblique rotation to yield $\widehat{\boldsymbol{\Lambda}}$ and $\widehat{\boldsymbol{\Phi}}_0$
  - Obtain estimates $\widehat{\boldsymbol{\Phi}}_\ell, \ell = 1, 2, \ldots$ by ordinary least squares.
  - Estimate the $\boldsymbol{A}_i$, $\boldsymbol{B}_j$, and $\boldsymbol{\Psi}$ by applying a generalization to multivariate time series of methods suggested by Box & Jenkins (1976, Appendix 6.2) to the $\widehat{\boldsymbol{\Phi}}_\ell$.
- Excellent approximation when $q = 0$

An Example

- Data from Lebo & Nesselroade (1978).
- Each of four female subjects rated her mood on six adjective rating scales daily for $T = 103$ days.
- Scales were Active, Lively, Peppy to define an Energy factor and Sluggish, Tired, Weary to define a Fatigue factor.
- Subject #4 is employed. For illustrative purposes an ARMA(1,1) model is used for the latent time series.
  - No. of variables, $k = 6$
  - No. of factors, $m = 2$
  - Maximum lag employed, $L = 3$

Lebo #4 Example: Exploratory PFA(1,1)

Starting values for parameter estimates

$\widehat{\boldsymbol{\Lambda}}$		$\widehat{\boldsymbol{\epsilon}}$
0.92	0.00	0.16
0.81	−0.17	0.14
0.72	−0.29	0.13
−0.14	0.71	0.35
0.00	0.93	0.14
−0.14	0.77	0.26

$\widehat{\boldsymbol{A}}_1$	
0.47	− 0.29
0.44	0.87
$\lvert\lambda_{\text{max}}\rvert = 0.73$	

$\widehat{\boldsymbol{B}}_1$	
− 0.15	0.53
− 0.35	− 0.53
$\lvert\lambda_{\text{max}}\rvert = 0.51$	

Shock correlations and sd's obtained from $\widehat{\boldsymbol{\Psi}}$

	1.00	− 0.73
	− 0.73	1.00
Sd	0.95	0.89

Figures in red represent parameter values fixed for identification purposes.

Lebo #4 Example: Exploratory PFA(1,1)

Functions of starting values

$\widehat{\boldsymbol{\Phi}}_0$		$\widehat{\boldsymbol{\Phi}}_2$		RSS	
1.00	−0.65	0.13	−0.15	$\widehat{F}_0$	0.00
−0.65	1.00	−0.01	0.29	$\widehat{F}_1$	0.13
				$\widehat{F}_2$	0.14
$\widehat{\boldsymbol{\Phi}}_1$		$\widehat{\boldsymbol{\Phi}}_3$		$\widehat{F}_3$	0.30
0.20	−0.09	0.06	−0.16	$\widehat{F}_T$	0.57
−0.11	0.38	0.05	0.18		

Figures in red represent function values that are constrained for identification purposes.

Lebo #4 Example: Exploratory PFA(1,1)

OLS solution

$\widehat{\boldsymbol{\Lambda}}$		$\widehat{\boldsymbol{\epsilon}}$
0.94	0.00	0.11
0.69	-0.29	0.18
0.70	-0.33	0.11
-0.16	0.64	0.43
0.00	0.90	0.19
0.02	0.94	0.15

$\widehat{\boldsymbol{A}}_1$	
0.73	0.12
0.06	0.85
$\lvert\lambda_{\max}\rvert = 0.89$	

$\widehat{\boldsymbol{B}}_1$	
-0.64	-0.22
0.10	-0.48
$\lvert\lambda_{\max}\rvert = 0.58$	

Shock correlations and sd's obtained from $\widehat{\boldsymbol{\Psi}}$

	1.00	-0.74
	-0.74	1.00
Sd	0.97	0.89

Lebo #4 Example: Exploratory PFA(1,1)

OLS solution

$$\widehat{\mathbf{\Phi}}_0 = \begin{array}{cc} 1.00 & -0.65 \\ -0.65 & 1.00 \end{array} \qquad \widehat{\mathbf{\Phi}}_2 = \begin{array}{cc} 0.12 & -0.05 \\ -0.07 & 0.30 \end{array}$$

$$\widehat{\mathbf{\Phi}}_1 = \begin{array}{cc} 0.18 & -0.12 \\ -0.09 & 0.36 \end{array} \qquad \widehat{\mathbf{\Phi}}_3 = \begin{array}{cc} 0.08 & 0.00 \\ -0.05 & 0.26 \end{array}$$

	RSS	
	OLS	Start
$\widehat{F}_0$	0.02	0.00
$\widehat{F}_1$	0.10	0.13
$\widehat{F}_2$	0.18	0.14
$\widehat{F}_3$	0.06	0.30
$\widehat{F}_T$	0.36	0.57

Lebo #4 Example: Exploratory PFA(1,1)

Oblique CF-Varimax Rotation

$\widehat{\boldsymbol{\Lambda}}$		$\widehat{\boldsymbol{\epsilon}}$
0.96	0.03	0.11
0.73	−0.25	0.18
0.74	−0.29	0.11
−0.23	0.59	0.43
−0.09	0.84	0.19
−0.07	0.88	0.15

$\widehat{\boldsymbol{A}}_1$	
0.71	0.10
0.05	0.86
$\lvert\lambda_{\text{max}}\rvert = 0.89$	

$\widehat{\boldsymbol{B}}_1$	
−0.63	−0.21
0.10	−0.50
$\lvert\lambda_{\text{max}}\rvert = 0.58$	

	Shock Corr.	
	1.00	−0.70
	−0.70	1.00
Sd	0.97	0.88

Lebo #4 Example: Exploratory PFA(1,1)

Oblique CF-Varimax Rotation

$\widehat{\mathbf{\Phi}}_0$		$\widehat{\mathbf{\Phi}}_2$		RSS	
1.00	−0.61	0.12	−0.04	$\widehat{F}_0$	0.02
−0.61	1.00	−0.07	0.33	$\widehat{F}_1$	0.10
				$\widehat{F}_2$	0.18
$\widehat{\mathbf{\Phi}}_1$		$\widehat{\mathbf{\Phi}}_3$		$\widehat{F}_3$	0.06
0.18	−0.12	0.08	0.00	$\widehat{F}_T$	0.36
−0.09	0.39	−0.05	0.28		

Lebo #4 Example: Exploratory PFA(1,1)

Alternative representation of ARMA process: MA Weights are replaced by shocks that are correlated across time.

Lagged shock correlations and standard deviations obtained from $\widehat{\boldsymbol{\Psi}}$ and $\widehat{\boldsymbol{B}}_1$.

Lag 0		Lag 1		Sd
1.00	-0.68	-0.39	0.20	1.09
-0.68	1.00	0.35	-0.43	1.02

Lebo #4 Example: Exploratory PFA(1,1)

Shock Factor analysis representation of PFA(1,1)

$\widehat{\mathbf{\Lambda}}_0^*$		$\widehat{\mathbf{\Lambda}}_1^*$		$\widehat{\mathbf{\Lambda}}_2^*$		$\widehat{\mathbf{\Lambda}}_3^*$	
0.93	0.03	0.08	-0.08	0.08	-0.02	0.07	0.01
0.71	-0.22	0.03	-0.15	0.02	-0.09	0.02	-0.06
0.72	-0.25	0.02	-0.16	0.02	-0.10	0.02	-0.06
-0.22	0.52	0.07	0.21	0.06	0.17	0.05	0.14
-0.08	0.74	0.12	0.28	0.10	0.23	0.09	0.20
-0.07	0.77	0.12	0.29	0.11	0.24	0.10	0.21

Note that the contemporaneous correlation of the two factors is -0.70

The largest absolute difference between process- and shock-implied correlations at $q^* = 3$ is -0.06

Lebo #4 Example: Exploratory PFA(1,1)

Alternative identification conditions

$\widehat{\Lambda}$		$\widehat{\epsilon}$
0.10	-0.94	0.11
-0.12	-0.90	0.18
-0.15	-0.93	0.11
0.42	0.63	0.43
0.62	0.66	0.19
0.64	0.67	0.15

$\widehat{A}_1$	
0.92	0.12
-0.06	0.65
$\lvert\lambda_{\text{max}}\rvert = 0.89$	

$\widehat{B}_1$	
-0.63	-0.13
0.15	-0.50
$\lvert\lambda_{\text{max}}\rvert = 0.58$	

$\widehat{\Psi}$	
0.62	0.00
0.00	0.95

$\widehat{\boldsymbol{\Phi}}_0$	
1.00	0.00
0.00	1.00

Note poor simple structure

Lebo #4 Example: Exploratory PFA(1,1)

Alternative identification conditions

$\widehat{\mathbf{\Phi}}_0$		$\widehat{\mathbf{\Phi}}_2$		RSS	
1.00	0.00	0.50	0.02	$\widehat{F}_0$	0.02
0.00	1.00	−0.01	0.12	$\widehat{F}_1$	0.10
				$\widehat{F}_2$	0.18
$\widehat{\mathbf{\Phi}}_1$		$\widehat{\mathbf{\Phi}}_3$		$\widehat{F}_3$	0.06
0.53	−0.00	0.46	0.03	$\widehat{F}_T$	0.36
0.04	0.18	−0.03	0.08		

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Abstract

The development of methodology for the factor analysis of data from a single subject over repeated observations on a battery of variables arose out of the P-technique recommended by Cattell. In this approach the data are treated as if the repeated observations are independent and subjected to a standard factor analysis. Comments by T. W. Anderson that the P-technique should be supplemented by the modeling of time dependence in the data were influential on later modifications of Cattell's approach.

Work on the P-technique is first reviewed and associated developments are presented. It is seen that two types of model have been used. In one, the process factor analysis model, the factors follow a VARMA process. This model has a single factor matrix and additional time series parameter matrices. In the other, the shock factor analysis model, the factors are independently distributed random shock vectors or, equivalently, white noise. This model has several factor matrices, each representing the effect of the latent shock vector on the vector of manifest variables at one of a number of lags. It has no time series parameter matrices. Different perspectives on the same situation are provided by the two models and it is instructive to employ them concurrently.

Two classes of estimation methodology are also considered. One uses full information maximum likelihood based on the original observations. The other is a moment-based approach. It involves the initial computation of lagged correlation matrices, followed by the fitting of appropriate correlation structures using a minimum discrepancy approach.

A modern development is then described. The process factor analysis model is estimated using a moment-based approach. Both confirmatory, and exploratory variants with rotation of factors, are available. In addition, the shock factor representation for any process factor analysis is provided. Examples are employed to illustrate capabilities of the model.